

Temporal discretisation of the Skyrme Model

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I. INTRODUCTION

Throughout this paper *CT* and *DT* refer to *continuous time* and *discrete time* respectively, units are taken as $c = \hbar = 1$ and the metric tensor has components $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. The symbol $=_c$ denotes an equality holding modulo the equations of motion. In recent years there has been much interest in the discretization of time for a variety of reasons. These include the study of **integrable systems** [1], motivated by the question of the integrability of various dynamical models. Systems studied are often point particle models with special properties which make the process of discretization successful, such as the Toda lattice [2]; **field theories** [3-4], considered as approximations to CT field theories. Generally these tend to look like variants of lattice gauge theory, with the important difference that time is not Wick rotated; **lattice gauge theory**, which has as its objective the CT limit $a \rightarrow 0$, where a is the lattice spacing and which is regarded as the place where physics occurs. The value $a \neq 0$ is taken only as a calculational device, suitable for providing a regularization procedure and for computer simulation to some given order of numerical accuracy. In this scheme discretization is also applied to the spatial coordinates, and because time is Euclidean, the theory uses a four dimensional Euclidean space. On a more fundamental level it has been speculated by many authors that perhaps space and/or time are not continuous on very short scales. In the history of quantum field theory, problems associated with operator products at the same spacetime point were sometimes addressed by “point-splitting” techniques, wherein two or more field operators at the same point were defined at separate points and only after the calculations was the separation taken to zero. This was a technique used for example by Schwinger in his famous 2-dimensional model of QED [5]. More recently there has been a strong suspicion that at the Planck scale the usual view of spacetime breaks down and novel new ideas should be contemplated. In a series of papers [3-6] attention was focussed on the consequences of discretization of time but not of space. Such an idea was discussed by 't Hooft in $2 + 1$ dimensions [10], with the suggestion that under certain circumstances, there may be variants of the Regge calculus approach to General relativity [11] wherein time becomes discrete but space does not. More recently, discrete time has been discussed as a direct consequence of quantum processes in action [12]. In this approach, time jumps whenever a factor state in the quantum state of the Universe undergoes a test resulting in information gain. These factor states can in principle include attributes such as momentum, so there is no necessity to discuss discrete space as well, although that is possible. In this article only temporal discretization will be discussed. In the system of DT mechanics discussed here, it is considered an exact form of mechanics, with its own consistent laws of motion and dynamical invariants, and not only in the CT limit. This is a fundamental difference between this DT mechanics and conventional lattice gauge theory. All of the operators in our DT quantum theory have to be good, in the language of lattice gauge theory. It is the case that *any* conventional continuous time theory can be discretized, simply by looking at the system at discrete times. For example, integrating the Lagrangian between chosen times (temporal nodes) gives Hamilton’s principal function (the Hamilton-Jacobi function), which can be regarded as a convenient discretization of time, because it depends only on what the system is doing (the co-ordinates) at the end points of the temporal interval. Our original motivation for discretizing time arose when computer simulation of a soliton theory was undertaken [13]. In CT theory, various integrals such as total mass, linear and angular momentum and charge remain invariants under dynamical evolution. However, it was soon found that a naive discretization of the equations of motion did not lead to the naive discretizations of these integrals remaining invariant during the simulations. This has nothing to do with numerical inaccuracy, but with the principles being employed. It was perceived that discretization should perhaps be done in a more carefully controlled, principled way, so that a genuine discrete dynamics with true invariants emerged. In this paper some of the methodology discussed in [7] is applied to a well-known field theoretic model with an underlying non-abelian group structure, the Skyrme model. This is not a trivial application, for it is not obvious at all that every model in CT field theory carries over easily into a DT analogue theory with DT analogues of the CT structures remaining intact. An important difference is that there may be no obvious DT analogue of the CT

Hamiltonian. It is the merit of integrable systems that such an analogue should be found, but in general this will not be so. The obvious reason is that by definition, there is no continuous translation in time in DT mechanics, so there is no DT analogue of the CT generator of such a transformation. Fortunately, the basic idea of a Noether invariant survives in DT mechanics. If there is a continuous symmetry of the system function (the DT analogue of the CT Lagrangian) then there will be a corresponding invariant, referred to here as a Maeda-Noether invariant [14]. In the DT Skyrme model, the symmetry which generates the axial and vector charges survives and the corresponding DT charges are discussed here. We shall also discuss T.D. Lee's version of DT mechanics, which automatically gives a DT analogue of energy.

II. DT MECHANICS

In the models discussed here the dynamical variables are defined only at instants of time nT where n is an integer and T is some fixed time interval. In these models there are no gauge fields defined on the temporal links connecting successive instants of time, so the formalism presented here takes on a simpler form than would be the case in say QED or QCD. The dynamical variables here are real valued fields φ^α , $\alpha = 0, 1, 2, 3$ with suitable constraints. The index α is not a Lorentz index, but it is lowered with the metric tensor components $\eta_{\mu\nu}$. The DT analogue of the CT Lagrange density $\mathcal{L} = \mathcal{L}(\varphi^\alpha, \partial_\mu \varphi^\alpha)$ is called the *system function density* and takes the form

$$S^n \equiv \mathcal{S}(\varphi_n^\alpha, \varphi_{n+1}^\alpha, \nabla \varphi_n^\alpha, \nabla \varphi_{n+1}^\alpha). \quad (1)$$

It has the physical dimensions of an action density, not an energy density. The integral $S^n \equiv \int d\mathbf{x} S^n$ is called the *system function*, and although used as the DT analogue of the Lagrangian L in CT mechanics, is more like a Hamilton's principal function. Applying Hamilton's principle to the action sum

$$A_{MN}[\Gamma] = \sum_{n=M}^{N-1} S^n \quad (2)$$

gives the second order equations of motion [7]

$$\Pi_n^\alpha = {}_c \bar{\Pi}_n^\alpha, \quad M < n < N-1 \quad (3)$$

where

$$\Pi_n^\alpha \equiv -\frac{\partial S^n}{\partial \varphi_n^\alpha} + \nabla \cdot \frac{\partial S^n}{\partial \nabla \varphi_n^\alpha}, \quad \bar{\Pi}_n^\alpha \equiv \frac{\partial S^{n-1}}{\partial \varphi_n^\alpha} - \nabla \cdot \frac{\partial S^{n-1}}{\partial \nabla \varphi_n^\alpha}. \quad (4)$$

Given an infinitesimal transformation of the fields $\varphi^\alpha \rightarrow \varphi^\alpha + \delta \varphi^\alpha$ then

$$\delta S^n = {}_c \delta \varphi_{n+1}^\alpha \bar{\Pi}_{n+1}^\alpha - \delta \varphi_n^\alpha \Pi_n^\alpha + \nabla \cdot \sigma, \quad (5)$$

where

$$\delta \sigma_n \equiv \delta \varphi_n^\alpha \frac{\partial S^n}{\partial \nabla \varphi_n^\alpha} + \delta \varphi_{n+1}^\alpha \frac{\partial S^n}{\partial \nabla \varphi_{n+1}^\alpha}. \quad (6)$$

A symmetry of the system function is a transformation of the fields which leaves the system function unchanged. Hence for such a symmetry

$$\begin{aligned} & \delta \varphi_{n+1}^\alpha \bar{\Pi}_{n+1}^\alpha - \delta \varphi_n^\alpha \Pi_n^\alpha + \nabla \cdot \delta \sigma_n = 0, \\ \Rightarrow & \delta \varphi_{n+1}^\alpha \Pi_{n+1}^\alpha - \delta \varphi_n^\alpha \Pi_n^\alpha + \nabla \cdot \delta \sigma_n = {}_c 0, \end{aligned} \quad (7)$$

which can be written in the forwards form

$$D_n^+ \delta \rho_n + \nabla \cdot \delta \mathbf{j}_n = {}_c 0 \quad (8)$$

using the equations of motion (3). Here the forwards densities are defined by

$$\delta \rho_n \equiv \delta \varphi_n^\alpha \Pi_n^\alpha, \quad \delta \mathbf{j}_n \equiv T^{-1} \delta \sigma_n, \quad (9)$$

with $D_n^+ \equiv (U_n - 1)/T$, where U_n is the temporal step operator defined by $U_n f_n = f_{n+1}$. From (8) the corresponding dynamical invariant is given by

$$\delta C^m \equiv \int \delta \varphi_n^\alpha \Pi_n^\alpha, \quad (10)$$

assuming that the fields fall off sufficiently rapidly at spatial infinity. An alternative form is given by noting that

$$\delta \varphi_{n+1}^\alpha \bar{\Pi}_{n+1}^\alpha - \delta \varphi_n^\alpha \bar{\Pi}_n^\alpha + \nabla \cdot \delta \sigma_n =_c 0, \quad (11)$$

giving the backwards DT equation of continuity

$$D_n^- \delta \bar{\rho}_n + \nabla \cdot \delta \mathbf{j}_n =_c 0 \quad (12)$$

where $D_n^- \equiv (1 - U_n^{-1})/T$ and the backwards densities are defined by

$$\delta \bar{\rho}_n \equiv \delta \varphi_{n+1}^\alpha \bar{\Pi}_{n+1}^\alpha, \quad \delta \mathbf{j}_n \equiv T^{-1} \delta \sigma_n. \quad (13)$$

III. THE SKYRME MODEL

The Skyrme model [15] has soliton solutions with a number of properties suggestive of baryon physics. Its basic dynamical degrees of freedom are space-time fields $U(x)$ which take values in $SU(2)$. Because this implies constraints, it is often convenient to parametrize these fields in terms of an unconstrained isotopic triplet of real scalar fields $\pi \equiv (\pi^1, \pi^2, \pi^3)$:

$$U \equiv \exp \{i \tau \cdot \pi\} = \cos(|\pi|) + i \frac{\tau \cdot \pi}{|\pi|} \sin(|\pi|), \quad (14)$$

where $\tau \equiv (\tau^1, \tau^2, \tau^3)$ are the Pauli matrices. With the definitions

$$U_\mu \equiv \partial_\mu U, \quad L_\mu \equiv U^+ \partial_\mu U = -\partial_\mu U^+ U \quad (15)$$

the Lagrange density may be written in the form

$$\mathcal{L}_s \equiv -\frac{F_\pi^2}{16} \text{Tr} L_\mu L^\mu - \frac{1}{32e^2} \text{Tr} [L_\mu, L_\nu] [L^\mu, L^\nu] \quad (16)$$

where F_π is the pion coupling constant and the second term is known as the *Skyrme term*. In terms of the U fields this is equivalent to

$$\mathcal{L}_s = \frac{F_\pi^2}{16} \text{Tr} U_\mu^+ U^\mu - \frac{1}{16e^2} \text{Tr} \{U_\mu^+ U_\nu U^{+\mu} U^\nu - U_\mu^+ U_\nu U^{+\nu} U^\mu\}. \quad (17)$$

In addition to the standard Poincaré symmetries an important symmetry of this Lagrangian is invariance under separate left and right $SU(2)$ transformations;

$$U \rightarrow U' \equiv A U B^+ \quad (18)$$

where A and B are spacetime independent elements of $SU(2)$. This generates the so-called axial and vector charges. In Appendix A the quaternion approach to the parametrization of the U variables is given. With this, U may be written in the form

$$U = q_\mu \varphi^\mu = \varphi^0 + i \tau^i \varphi^i, \quad (19)$$

where $q_0 \equiv I_2$, the 2×2 identity matrix and $q_i \equiv i \tau^i$, $i = 1, 2, 3$ have the properties of the quaternions i, j, k . The four real fields φ^μ are read off from (14) to be

$$\varphi^0 \equiv \cos(|\pi|), \quad \varphi^i \equiv \sin(|\pi|) n^i, \quad \mathbf{n} \cdot \mathbf{n} = 1. \quad (20)$$

Since there are only three independent parameters describing the elements of $SU(2)$, the four components φ^μ are constrained to the surface of S^3 , the unit sphere in four dimensions, i.e.

$$\varphi^\mu \varphi^\mu = 1. \quad (21)$$

With this reparametrisation the Lagrange density becomes

$$\mathcal{L}_s = \frac{\alpha^2}{2} \partial_\mu \varphi^\alpha \partial^\mu \varphi^\alpha - \frac{\beta^2}{4} \partial_\mu \varphi^\alpha \partial_\nu \varphi^\beta \{ \partial^\mu \varphi^\beta \partial^\nu \varphi^\alpha - \partial^\mu \varphi^\alpha \partial^\nu \varphi^\beta \} + \frac{1}{2} \mu (\varphi^\alpha \varphi^\alpha - 1) \quad (22)$$

where $\alpha^2 \equiv \frac{1}{4} F_\pi^2$, $\beta^2 \equiv e^{-2}$ and the Lagrange multiplier μ enforces the S^3 constraint (21). Then the conjugate momenta are given by

$$\pi^\alpha \equiv \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^\alpha} = M_{\alpha\beta} \dot{\varphi}^\beta \quad (23)$$

where

$$M_{\alpha\beta} = (\alpha^2 - \beta^2 \partial_i \varphi^\mu \partial_i \varphi^\mu) \delta_{\alpha\beta} + \beta^2 \partial_i \varphi^\alpha \partial_i \varphi^\beta. \quad (24)$$

The constraints turn out to be second class in the terminology of Dirac [16] and given by

$$\chi_1 \equiv \varphi^\alpha \varphi^\alpha - 1 \approx 0, \quad \chi_2 \equiv \varphi^\alpha \pi^\alpha \approx 0 \quad (25)$$

Then the non-zero Dirac brackets are evaluated to be

$$\begin{aligned} \{ \pi_{\mathbf{x}}^\alpha, \varphi_{\mathbf{y}}^\beta \}_{DB} &= (\varphi_{\mathbf{x}}^\alpha \varphi_{\mathbf{x}}^\beta - \delta_{\alpha\beta}) \delta^3(\mathbf{x} - \mathbf{y}), \\ \{ \pi_{\mathbf{x}}^\alpha, \pi_{\mathbf{y}}^\beta \}_{DB} &= (\pi_{\mathbf{x}}^\alpha \varphi_{\mathbf{x}}^\beta - \pi_{\mathbf{x}}^\beta \varphi_{\mathbf{x}}^\alpha) \delta^3(\mathbf{x} - \mathbf{y}). \end{aligned} \quad (26)$$

It is these which should be used in the quantization of the fields.

IV. THE $SU(2)$ PARTICLE

A. continuous time

In this section the most basic variant of the Skyrme model is considered, which is to drop the Skyrme term and the spatial degrees of freedom. Then the Lagrange density reduces to the Lagrangian

$$L = \frac{1}{4} \alpha^2 \text{Tr} \dot{U}^+ \dot{U}, \quad (27)$$

where $U \equiv U(t)$ is a time dependent element of $SU(2)$ and $\alpha = \frac{1}{2} F_\pi$. The number of independent real dynamical variables is three and there are two alternative formulations:

1. the π fields

The U fields may be parametrized using three unconstrained real fields: $U(t) = \exp\{i\tau \cdot \pi(t)\}$, where $\pi(t) \equiv F(t) \mathbf{n}(t)$ is an element of \mathcal{R}^3 and $\mathbf{n}(t)$ is a unit 3-vector. Then

$$U(t) = \cos F + i \sin F \tau \cdot \mathbf{n}. \quad (28)$$

The mapping from the \mathcal{R}^3 space of the parameters π to $SU(2)$ is many to one, with the vectors $(F + 2k\pi) \mathbf{n}$, k an integer, mapping into the same point of $SU(2)$. The Lagrangian (27) then reduces to

$$L = \frac{1}{2} \alpha^2 \left\{ \dot{F}^2 + \dot{\mathbf{n}} \cdot \dot{\mathbf{n}} \sin^2 F \right\} + \frac{1}{2} \mu (\mathbf{n} \cdot \mathbf{n} - 1), \quad (29)$$

where a Lagrange multiplier is included to enforce the normalization condition on the unit vector \mathbf{n} . The equations of motion are

$$\ddot{F} - \sin F \cos F \dot{\mathbf{n}} \cdot \dot{\mathbf{n}} =_c 0, \quad (30)$$

$$\alpha^2 \sin^2 F \ddot{\mathbf{n}} + 2\alpha^2 \sin F \cos F \dot{F} \dot{\mathbf{n}} =_c \mu \mathbf{n}, \quad \mathbf{n} \cdot \mathbf{n} =_c 1. \quad (31)$$

In phase space the system has two second class constraints in the language of Dirac [16]. Now define p, \mathbf{p} to be the momenta conjugate to F and \mathbf{n} respectively. Then these constraints take the form

$$\chi_1 \equiv \mathbf{n} \cdot \mathbf{n} - 1 \approx 1, \quad \chi_2 \equiv \mathbf{n} \cdot \mathbf{p} \approx 0. \quad (32)$$

Following Dirac [16], the Dirac brackets can be constructed in the standard way giving the non-zero brackets

$$\{p, F\}_D = -1, \quad \{p^i, n^j\}_D = -\delta_{ij} + n^i n^j, \quad \{p^i, p^j\}_D = p^i n^j - p^j n^i. \quad (33)$$

The total Hamiltonian is given by

$$H_T = \frac{p^2}{2\alpha^2} + \frac{\mathbf{p} \cdot \mathbf{p}}{2\alpha^2 \sin^2 F}, \quad (34)$$

which is an invariant of the motion. Two additional invariants of the motion can be found using Noether's theorem by observing that the transformation

$$U \rightarrow U' \equiv AUB^+ \quad (35)$$

is a symmetry of the Lagrangian, where A and B are space and time independent elements of $SU(2)$. Writing $A \simeq 1 + i\tau \cdot \mathbf{a}$, $B \simeq 1 + i\tau \cdot \mathbf{b}$, where \mathbf{a} and \mathbf{b} are infinitesimal then

$$\delta F = (\mathbf{a} - \mathbf{b}) \cdot \mathbf{n}, \quad \delta \mathbf{n} = \cot F \{ \mathbf{a} - \mathbf{b} - \mathbf{n} \cdot (\mathbf{a} - \mathbf{b}) \mathbf{n} \} + \mathbf{n} \times (\mathbf{a} + \mathbf{b}) \quad (36)$$

to lowest order in the infinitesimal parameters. Now an application of Noether's theorem gives the conserved left and right charges

$$\mathbf{L} \equiv \mathbf{n} \times \mathbf{p} - p\mathbf{n} - \cot F \mathbf{p}, \quad \mathbf{R} \equiv \mathbf{n} \times \mathbf{p} + p\mathbf{n} + \cot F \mathbf{p} \quad (37)$$

in phase-space. In configuration space they take the form

$$\begin{aligned} \mathbf{L} &= \alpha^2 \left\{ \sin^2 F \mathbf{n} \times \dot{\mathbf{n}} - \dot{F} \mathbf{n} - \cos F \sin F \dot{\mathbf{n}} \right\} \equiv \mathbf{A} - \mathbf{V} \\ \mathbf{R} &= \alpha^2 \left\{ \sin^2 F \mathbf{n} \times \dot{\mathbf{n}} + \dot{F} \mathbf{n} + \cos F \sin F \dot{\mathbf{n}} \right\} \equiv \mathbf{A} + \mathbf{V} \end{aligned} \quad (38)$$

where

$$\mathbf{A} \equiv \alpha^2 \sin^2 F \mathbf{n} \times \dot{\mathbf{n}}, \quad \mathbf{V} \equiv \alpha^2 \left(\dot{F} \mathbf{n} + \cos F \sin F \dot{\mathbf{n}} \right),$$

are conserved separately. These are known conventionally as the *vector* and *axial* charges respectively.

2. The φ fields

The Lagrangian takes the form

$$L \equiv \frac{1}{2} \alpha^2 \dot{\varphi}^\alpha \dot{\varphi}^\alpha + \frac{1}{2} \mu (\varphi^\alpha \varphi^\alpha - 1) \quad (39)$$

and gives equations of motion

$$\alpha^2 \ddot{\varphi}^\alpha =_c \mu \varphi^\alpha, \quad \varphi^\alpha \varphi^\alpha =_c 1, \quad (40)$$

i.e.

$$\ddot{\varphi}^\alpha =_c (\varphi^\beta \ddot{\varphi}^\beta) \varphi^\alpha. \quad (41)$$

The Lagrangian is invariant to the global $SU(2)$ transformation $U' = AUB^+$ where A and B are time independent elements of $SU(2)$. Now suppose

$$A \simeq 1 + i\tau \cdot \mathbf{a}, \quad B \simeq 1 + i\tau \cdot \mathbf{b} \quad (42)$$

where \mathbf{a} and \mathbf{b} are infinitesimal, then

$$\delta\varphi^0 = (\mathbf{b} - \mathbf{a}) \cdot \boldsymbol{\varphi}, \quad \delta\boldsymbol{\varphi} = (\mathbf{a} - \mathbf{b})\varphi^0 + \boldsymbol{\varphi} \times (\mathbf{a} + \mathbf{b}) \quad (43)$$

Hence the conserved charge is given by

$$\delta\rho = (\mathbf{a} - \mathbf{b}) \cdot \mathbf{V} + (\mathbf{a} + \mathbf{b}) \cdot \mathbf{A}$$

where

$$\mathbf{V} \equiv \alpha^2 (\varphi^0 \dot{\boldsymbol{\varphi}} - \dot{\varphi}^0 \boldsymbol{\varphi}), \quad \mathbf{A} \equiv \alpha^2 \dot{\boldsymbol{\varphi}} \times \boldsymbol{\varphi} \quad (44)$$

with $\dot{\mathbf{V}} =_c \dot{\mathbf{A}} =_c \mathbf{0}$.

B. temporal discretization

Turning to the temporal discretization of the $SU(2)$ particle system, the problem reduces to choosing a suitable virtual path between temporal nodes [6]. It was found that appropriate paths were of the form

$$U_{n\lambda} \equiv \lambda U_{n+1} + (1 - \lambda)U_n \quad (45)$$

where the parameter $\lambda \in [0, 1]$ interpolates temporal nodes and

$$U_n \equiv q_\alpha \varphi_n^\alpha, \quad \varphi_n^\alpha \varphi_n^\alpha = 1. \quad (46)$$

Given the Lagrangian

$$L \equiv \frac{1}{4} \alpha^2 \text{Tr} \dot{U}^\dagger \dot{U} \quad (47)$$

then the system function becomes

$$S^n = \frac{1}{2} T \alpha^2 D_n^+ \varphi_n^\alpha D_n^+ \varphi_n^\alpha + \frac{1}{4} \mu_n T (\varphi_n^\alpha \varphi_n^\alpha - 1) + \frac{1}{4} \mu_{n+1} T (\varphi_{n+1}^\alpha \varphi_{n+1}^\alpha - 1) \quad (48)$$

The equations of motion are

$$\varphi_{n+1}^\alpha - 2\varphi_n^\alpha + \varphi_{n-1}^\alpha =_c \mu_n T^2 \varphi_n^\alpha, \quad \varphi_n^\alpha \varphi_n^\alpha =_c 1 \quad (49)$$

This is equivalent to

$$\varphi_{n+1}^\alpha + \varphi_{n-1}^\alpha =_c \varphi_n^\beta (\varphi_{n+1}^\beta + \varphi_{n-1}^\beta) \varphi_n^\alpha \quad (50)$$

There is invariance under

$$U_n \rightarrow U'_n \equiv A U_n B^\dagger, \quad A, B \in SU(2) \quad (51)$$

Then if

$$A \simeq 1 + i\tau \cdot \mathbf{a} \quad B \simeq 1 + i\tau \cdot \mathbf{b} \quad (52)$$

then

$$\delta\varphi_n^0 = -(\mathbf{a} - \mathbf{b}) \cdot \boldsymbol{\varphi}_n, \quad \delta\boldsymbol{\varphi}_n = \varphi_n^0 (\mathbf{a} - \mathbf{b}) + \boldsymbol{\varphi}_n \times (\mathbf{a} + \mathbf{b}) \quad (53)$$

Then

$$\delta\rho_n \equiv \delta\varphi_n^\alpha \Pi_n^\alpha = (\mathbf{a} - \mathbf{b}) \cdot \mathbf{V}^n + (\mathbf{a} + \mathbf{b}) \cdot \mathbf{A}^n, \quad (54)$$

giving the conserved charges

$$\begin{aligned}\mathbf{A}^n &\equiv \alpha^2 D_n^+ \boldsymbol{\varphi}_n \times \boldsymbol{\varphi}_n = \frac{\alpha^2}{T} \boldsymbol{\varphi}_{n+1} \times \boldsymbol{\varphi}_n \\ \mathbf{V}^n &\equiv \alpha^2 [\varphi_n^0 D_n^+ \boldsymbol{\varphi}_n - D_n^+ \varphi_n^0 \boldsymbol{\varphi}_n] = \frac{\alpha^2}{T} (\varphi_n^0 \boldsymbol{\varphi}_{n+1} - \varphi_{n+1}^0 \boldsymbol{\varphi}_n)\end{aligned}\quad (55)$$

If these charges are non-zero, then there is a natural time independent frame in isospace given by the directions $(\mathbf{V}^n, \mathbf{A}^n, \mathbf{A}^n \times \mathbf{V}^n)$. The following argument simplifies the equations of motion and establishes the existence of an infinite hierarchy of quadratic invariants. First note that, regardless of the equations of motion, the quantity $U_{n+1}^+ U_n$ is an element of $SU(2)$, and so may be written in the form

$$U_{n+1}^+ U_n = q_\lambda \Phi_n^\lambda \quad (56)$$

using the quaternionic notation discussed in Appendix A, where

$$\Phi_n^\lambda \equiv c^{\alpha\beta} {}_\lambda \varphi_{n+1}^\alpha \varphi_n^\beta \quad (57)$$

and

$$\Phi_n^\lambda \Phi_n^\lambda = 1. \quad (58)$$

In detail, the components of Φ^λ turn out to be

$$\Phi_n^0 = \varphi_n^\alpha \varphi_{n+1}^\alpha, \quad \boldsymbol{\Phi}_n = \mathbf{v}^n - \mathbf{a}^n, \quad (59)$$

where

$$\mathbf{v}^n \equiv \frac{T}{\alpha^2} \mathbf{V}^n = \varphi_n^0 \boldsymbol{\varphi}_{n+1} - \varphi_{n+1}^0 \boldsymbol{\varphi}_n, \quad \mathbf{a}^n \equiv \frac{T}{\alpha^2} \mathbf{A}^n = \boldsymbol{\varphi}_{n+1} \times \boldsymbol{\varphi}_n. \quad (60)$$

From this it follows that $(\Phi_n^0)^2$ is an invariant of the motion, namely

$$(\Phi_n^0)^2 = {}_c (\Phi_{n-1}^0)^2, \quad (61)$$

so we may write

$$\varphi_n^\alpha \varphi_{n+1}^\alpha = {}_c C \varepsilon_n, \quad (62)$$

for some real constant C and where $\varepsilon_n = \pm 1$. It turns out that

$$C^2 + \mathbf{v}_n^2 + \mathbf{a}_n^2 = 1, \quad (63)$$

which means that $-1 \leq C \leq 1$. Hence the equation of motion can be written in the form

$$\varphi_{n+1}^\alpha + \varphi_{n-1}^\alpha = {}_c C (\varepsilon_n + \varepsilon_{n-1}) \varphi_n^\alpha, \quad (64)$$

where the ε_n are of magnitude +1 but otherwise arbitrary. This arbitrariness can be traced to the use of the Lagrange multipliers μ_n in the system function (48) and is not a feature that exists in the CT limit $T \rightarrow 0$. In the special case that $\varepsilon_n = +1 \forall n$ then the equation is recognized to be equivalent to the DT harmonic oscillator discussed in [6]. Moreover, the bounds on the constant C mean that the motion is never hyperbolic. In this case it is found that

$$\varphi_n^\alpha \varphi_n^\alpha = 1, \quad \varphi_{n+1}^\alpha \varphi_n^\alpha = C, \quad \varphi_{n+2}^\alpha \varphi_n^\alpha = 2C^2 - 1$$

and generalizing this result gives the infinite set of invariants

$$\varphi_{n+m}^\alpha \varphi_n^\alpha = T_m(C)$$

where T_m is a Chebyshev polynomial of Type 1 [17] In terms of the F, \mathbf{n} description, the parametrization is given by

$$U_n = c_n + i s_n \mathbf{n}_n \quad (65)$$

where $c_n \equiv \cos F_n$, $s_n \equiv \sin F_n$, $\mathbf{n}_n \cdot \mathbf{n}_n = 1$, and then

$$\mathbf{a}^n = s_n s_{n+1} \mathbf{n}_n \times \mathbf{n}_{n+1}, \quad \mathbf{v}^n = s_n c_{n+1} \mathbf{n}_n - s_{n+1} c_n \mathbf{n}_{n+1}. \quad (66)$$

A. continuous time

The model is now extended to include spatial dependence, but not the quartic terms in the original Lagrangian. We shall call this the σ model. The CT Lagrange density is now

$$\mathcal{L} = \frac{1}{4}\alpha^2 \text{Tr} \partial_\mu U^\dagger \partial^\mu U, \quad (67)$$

which is equivalent to

$$\mathcal{L} = \frac{1}{2}\alpha^2 \partial_\mu \varphi^\alpha \partial^\mu \varphi^\alpha + \frac{1}{2}\mu (\varphi^\alpha \varphi^\alpha - 1). \quad (68)$$

The equations of motion are

$$\alpha^2 \square \varphi^\alpha =_c \mu \varphi^\alpha, \quad \varphi^\alpha \varphi^\alpha =_c 1 \quad (69)$$

which reduce to

$$\square \varphi^\alpha =_c (\varphi^\beta \square \varphi^\beta) \varphi^\alpha. \quad (70)$$

The conserved energy-momentum tensor density is

$$\mathcal{T}^{\mu\nu} = \alpha^2 \left\{ \partial^\mu \varphi^\alpha \partial^\nu \varphi^\alpha - \frac{1}{2} \eta^{\mu\nu} \partial_\beta \varphi^\alpha \partial^\beta \varphi^\alpha \right\}. \quad (71)$$

The invariance of the Lagrange density to the same transformation as before allows the vector and axial currents to be determined. Under the infinitesimal transformation

$$U \rightarrow U' \equiv (1 + i\boldsymbol{\tau} \cdot \mathbf{a})U (1 - i\boldsymbol{\tau} \cdot \mathbf{b}) \quad (72)$$

then the fields change according to the rule

$$\delta \varphi^0 = (\mathbf{b} - \mathbf{a}) \cdot \boldsymbol{\varphi}, \quad \delta \boldsymbol{\varphi} = (\mathbf{a} - \mathbf{b}) \varphi^0 + \boldsymbol{\varphi} \times (\mathbf{a} + \mathbf{b}), \quad (73)$$

giving the conserved currents

$$\mathbf{V}^\mu = \varphi^0 \partial^\mu \boldsymbol{\varphi} - \partial^\mu \varphi^0 \boldsymbol{\varphi}, \quad \mathbf{A}^\mu = \partial^\mu \boldsymbol{\varphi} \times \boldsymbol{\varphi}. \quad (74)$$

B. discrete time

The system function density for the σ -model is taken to be

$$\begin{aligned} \mathcal{S}^n = & \frac{1}{2} T \alpha^2 \{ D_n^+ \varphi_n^\alpha D_n^+ \varphi_n^\alpha - \frac{1}{2} \nabla \varphi_n^\alpha \cdot \nabla \varphi_n^\alpha - \nabla \varphi_{n+1}^\alpha \cdot \nabla \varphi_{n+1}^\alpha \} \\ & + \frac{1}{4} T \mu_n (\varphi_n^\alpha \varphi_n^\alpha - 1) + \frac{1}{4} T \mu_{n+1} (\varphi_{n+1}^\alpha \varphi_{n+1}^\alpha - 1), \end{aligned} \quad (75)$$

which gives equation of motion

$$\frac{\alpha^2}{T} \{ \varphi_{n+1}^\alpha - 2\varphi_n^\alpha + \varphi_{n-1}^\alpha \} - T \alpha^2 \nabla^2 \varphi_n^\alpha =_c T \mu_n \varphi_n^\alpha, \quad \varphi_n^\alpha \varphi_n^\alpha =_c 1. \quad (76)$$

There is invariance of the system function density under the infinitesimal transformation (53), and this gives two conserved currents:

$$D_n^+ \mathbf{V}_n^0 + \partial_i \mathbf{V}_n^i =_c \mathbf{0}, \quad D_n^+ \mathbf{A}_n^0 + \partial_i \mathbf{A}_n^i =_c \mathbf{0}, \quad (77)$$

where

$$\begin{aligned}
\mathbf{V}_n^0 &\equiv \varphi_n^0 \mathbf{\Pi}_n - \varphi_n \mathbf{\Pi}_n^0, \\
\mathbf{V}_n^i &\equiv \frac{1}{T} \left\{ -\varphi_n \frac{\partial \mathcal{S}^n}{\partial \partial_i \varphi_n^0} - \varphi_{n+1} \frac{\partial \mathcal{S}^n}{\partial \partial_i \varphi_{n+1}^0} + \varphi_n^0 \frac{\partial \mathcal{S}^n}{\partial \partial_i \varphi_n} + \varphi_{n+1}^0 \frac{\partial \mathcal{S}^n}{\partial \partial_i \varphi_{n+1}} \right\} \\
\mathbf{A}_n^0 &\equiv \mathbf{\Pi}_n \times \varphi_n \\
\mathbf{A}_n^i &\equiv \frac{1}{T} \left\{ \frac{\partial \mathcal{S}^n}{\partial \partial_i \varphi_n} \times \varphi_n + \frac{\partial \mathcal{S}^n}{\partial \partial_i \varphi_{n+1}} \times \varphi_{n+1} \right\}
\end{aligned} \tag{78}$$

i.e

$$\begin{aligned}
\mathbf{V}_n^0 &\equiv \frac{\alpha^2}{T} \{ \varphi_n^0 \varphi_{n+1} - \varphi_n \varphi_{n+1}^0 \} + \frac{1}{2} T \alpha^2 \{ \varphi_n \nabla^2 \varphi_n^0 - \varphi_n^0 \nabla^2 \varphi_n \} \\
\mathbf{V}_n^i &\equiv \frac{1}{2} \alpha^2 \{ \varphi_n \partial_i \varphi_n^0 - \varphi_n^0 \partial_i \varphi_n + \varphi_{n+1} \partial_i \varphi_{n+1}^0 - \varphi_{n+1}^0 \partial_i \varphi_{n+1} \} \\
\mathbf{A}_n^0 &\equiv \frac{\alpha^2}{T} \varphi_{n+1} \times \varphi_n - \frac{1}{2} T \alpha^2 \nabla^2 \varphi_n \times \varphi_n \\
\mathbf{A}_n^i &\equiv -\frac{1}{2} \alpha^2 \partial_i \varphi_n \times \varphi_n - \frac{1}{2} \alpha^2 \partial_i \varphi_{n+1} \times \varphi_{n+1}.
\end{aligned} \tag{79}$$

VI. DISCRETIZATION OF THE FULL MODEL

We may rewrite the full Skyrme Lagrange density equation (22) in the form

$$\mathcal{L}_s = \frac{1}{2} \dot{\varphi}^\alpha M^{\alpha\beta} (\partial_i \varphi^\mu) \dot{\varphi}^\beta - W (\partial_i \varphi^\mu) + \frac{1}{2} \mu (\varphi^\alpha \varphi^\alpha - 1), \tag{80}$$

where $M^{\alpha\beta} (\partial_i \varphi^\mu)$ is given by equation (24) and the potential function $W (\partial_i \varphi^\mu)$ contains quadratic and quartic terms in the derivatives of the fields. A serious problem may arise if any term in a Lagrangian is greater than quadratic in the dynamical variables. It is possible in such a case that certain discretizations leads to implicit equations of motion which cannot be solved directly to give the future values of the variables from a knowledge of their previous values. This remark applies to a classical theory. It is interesting that quantization may improve the situation. Suppose we had a system with dynamical variable x_n , and the DT equation of motion was an implicit one of the form

$$\Phi (x_{n-1}, x_n, x_{n+1}) =_c 0, \quad n = 1, 2, \dots \tag{81}$$

where Φ is some function. If we could not solve this equation, viz, find a unique expression of the form

$$x_{n+1} =_c \Omega (x_{n-1}, x_n), \tag{82}$$

then the implication is that there exists more than one solution consistent with the initial conditions. Alternatively, given x_{n-1} and x_{n+1} , there would not be a unique DT trajectory (i.e., unique value of x_n) connecting these points. This is of course unsettling and contrary to classical principles, but not so in quantum theory. We just have to recall Feynman's path integral, which explicitly requires us to consider all possible trajectories between initial and final times, including non-classical ones. It will be seen from this that DT mechanics should accommodate quantum mechanics better than classical mechanics, except in simple cases such as the DT harmonic oscillator. The DT Feynman rules for a φ^3 scalar field theory was discussed in [8], and it was found that the momentum space rules had softened vertices, due to the effects of temporal discretization, which is a potentially useful result. Analogous effects are expected from non-commutative spacetime theories. There are several possible discretizations of the Lagrange density (80), such as

$$\begin{aligned}
\mathcal{S}^n &= \frac{1}{2} T D_n^+ \varphi_n^\alpha M^{\alpha\beta} \left(\frac{1}{2} \partial_i \varphi_n^\mu + \frac{1}{2} \partial_i \varphi_{n+1}^\mu \right) D_n^+ \varphi_n^\beta - T W \left(\frac{1}{2} \partial_i \varphi_n^\mu + \frac{1}{2} \partial_i \varphi_{n+1}^\mu \right) \\
&\quad + \frac{1}{4} T \mu (\varphi_n^\alpha \varphi_n^\alpha - 1) + \frac{1}{4} T \mu (\varphi_{n+1}^\alpha \varphi_{n+1}^\alpha - 1)
\end{aligned} \tag{83}$$

or

$$\begin{aligned}
\mathcal{S}^n &= \frac{1}{4} T D_n^+ \varphi_n^\alpha \{ M^{\alpha\beta} (\partial_i \varphi_n^\mu) + M^{\alpha\beta} (\partial_i \varphi_{n+1}^\mu) \} D_n^+ \varphi_n^\beta \\
&\quad - \frac{1}{2} T \{ W (\partial_i \varphi_n^\mu) + W (\partial_i \varphi_{n+1}^\mu) \} \\
&\quad + \frac{1}{4} T \mu (\varphi_n^\alpha \varphi_n^\alpha - 1) + \frac{1}{4} T \mu (\varphi_{n+1}^\alpha \varphi_{n+1}^\alpha - 1).
\end{aligned} \tag{84}$$

Whichever form is chosen, the (implicit) equation of motion will be given by formula (3). In any case, the symmetries discussed in previous sections will hold, because the variations are global. The conserved DT vector and axial charges can be readily worked out, and left as an exercise. There is now a guarantee that these are dynamical invariants, even though it may not be possible to put the equation of motion into explicit form.

VII. DISCRETE TIME ENERGY

The methods discussed above permit the construction of those invariants associated with continuous symmetries, such as linear and angular momentum, and various charges, but not energy. Fortunately, there is a way of extending the formalism to generate a DT analogue of energy. This was discussed by T.D. Lee [18]. The method is to take the time intervals $T_n \equiv t_{n+1} - t_n$ to be dynamical, i.e., subject to their own dynamical equations of motion. The first thing is to change the system function density according to the rule

$$\mathcal{S}^n \equiv \mathcal{S}(\varphi_n^\alpha, \varphi_{n+1}^\alpha, \nabla \varphi_n^\alpha, \nabla \varphi_{n+1}^\alpha, T) \rightarrow \mathcal{S}^n(T_n) \equiv \mathcal{S}(\varphi_n^\alpha, \varphi_{n+1}^\alpha, \nabla \varphi_n^\alpha, \nabla \varphi_{n+1}^\alpha, T_n), \quad (85)$$

where T_n is the temporal measure between nodes n and $n+1$. The action integral now becomes

$$A_{MN}[\Gamma] = \sum_{n=M}^{N-1} \mathcal{S}^n(T_n) - \lambda \left(\sum_{n=M}^{N-1} T_n - T_{MN} \right), \quad (86)$$

where the Lagrange multiplier enforces the constraint that the sum of the temporal measures adds up to the fixed, total time T_{MN} . The equations of motion for the ordinary dynamical variables follows exactly the same pattern as in the regular case (constant T), but now there are the additional equations

$$\frac{\partial}{\partial T_n} \mathcal{S}^n(T_n) = \lambda, \quad n = M, M+1, \dots, N-1. \quad (87)$$

This gives a guarantee that the object

$$C^n \equiv \frac{\partial}{\partial T_n} \mathcal{S}^n(T_n) \quad (88)$$

is a dynamical invariant. Actually, the correct way to see what is happening is to see the equality of the C^n as dynamical equations which the T_n must satisfy. In general, numerical analysis would be required to solve these extended equations, and this may be non-trivial. Formally, however, the problem of energy drift is completely solved by this technique. Finally, this approach does not cause a problem with other dynamical invariants, such as the vector and axial charges.

VIII. QUANTIZATION

Quantization in DT particle and field theories can be readily discussed using the Schwinger action principle, and it should be possible to extend this method to the Skyrme Lagrangian discussed here. This will be left for another occasion.

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X. APPENDIX A : QUATERNIONS

Given the Pauli algebra

$$\tau^i \tau^j = \delta_{ij} I_2 + i \epsilon_{ijk} \tau^k$$

where I_2 is the 2×2 identity matrix, define the quaternionic symbols

$$q^i \equiv -i\tau^i, \quad q^0 \equiv I_2. \quad (89)$$

These satisfy the quaternionic multiplication rule

$$q^i q^j = -\delta_{ij} q^0 + \epsilon_{ijk} q^k. \quad (90)$$

Now define upper and lower quaternion indices with conjugation as follows:

$$\begin{aligned} q^\mu &\equiv (q^0, q^i), & q_\mu &\equiv (q_0, q_i) \\ q_0 &\equiv q^0 & q_i &\equiv -q^i = (q^i)^*. \end{aligned} \quad (91)$$

Then conjugation is equivalent to the following quaternionic index raising and lowering action:

$$(q^\mu)^* = q_\mu, \quad (q_\mu)^* = q^\mu. \quad (92)$$

The product rule (90) may be written in the compact form

$$q^\mu q^\nu = c^{\mu\nu}_\lambda q^\lambda \quad (93)$$

where

$$\begin{aligned} c^{00}_0 &= 1, & c^{0i}_0 &= c^{i0}_0 = 0, & c^{ij}_0 &= -\delta_{ij} \\ c^{00}_i &= 0, & c^{0i}_j &= c^{i0}_j = \delta_{ij}, & c^{ij}_k &= \epsilon_{ijk}. \end{aligned} \quad (94)$$

Note that $c^{\mu\nu}_0 = \eta^{\mu\nu} = \eta_{\mu\nu}$, where $\eta^{\mu\nu}$ are the components of the Lorentz metric tensor. The product of three quaternions is given by

$$q^\mu q^\nu q^\alpha = c^{\mu\nu}_\lambda c^{\lambda\alpha}_\beta q^\beta \quad (95)$$

and similarly for higher powers of the quaternions. The Pauli matrices are 2×2 matrices and this permits us to define a linear mapping Tr , called the *trace*, on the quaternions to the reals defined by

$$\begin{aligned} Tr q^\mu &= 2\delta^\mu_0, & Tr q^\mu q^\nu &= 2\eta^{\mu\nu} \\ Tr q^0 q^i &= 0, & Tr q^i q^j &= -2\delta_{ij}. \end{aligned} \quad (96)$$

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